ECED 3300
Tutorial 1

Problem 1

Show that the vector components of $\mathbf{A}$ **parallel** *and* **perpendicular** *to a given vector* $\mathbf{a}_s$ *can be expressed as*

(a) $\mathbf{A}_{\|} = (\mathbf{A} \cdot \mathbf{a}_s)\mathbf{a}_s$;

(b) $\mathbf{A}_{\perp} = \mathbf{a}_s \times (\mathbf{A} \times \mathbf{a}_s)$.

Solution

(a) By definition of the scalar product, $|\mathbf{A}_{\|}| = (\mathbf{A} \cdot \mathbf{a}_s)$ as $\mathbf{a}_s$ is a unit vector. It follows at once that $\mathbf{A}_{\|} = |\mathbf{A}_{\|}|\mathbf{a}_s = (\mathbf{A} \cdot \mathbf{a}_s)\mathbf{a}_s$.

(b) Using the "bac-cab" formula, $\mathbf{a}_s \times (\mathbf{A} \times \mathbf{a}_s) = \mathbf{A}(\mathbf{a}_s \cdot \mathbf{a}_s) - \mathbf{a}_s(\mathbf{A} \cdot \mathbf{a}_s) = \mathbf{A} - \mathbf{A}_{\|} = \mathbf{A}_{\perp}$.

Problem 2

Given two vectors $\mathbf{A} = \alpha\mathbf{a}_x + 3\mathbf{a}_y - 2\mathbf{a}_z$ *and* $\mathbf{B} = 4\mathbf{a}_x + \beta\mathbf{a}_y + 8\mathbf{a}_z$*, determine* α *and* β *such that*

(a) $\mathbf{A}$ *and* $\mathbf{B}$ *are* **parallel***,*

(b) $\mathbf{A}$ *and* $\mathbf{B}$ *are* **perpendicular**.

Solution

(a) $\mathbf{A}$ and $\mathbf{B}$ are **parallel** implies that there is a constant k such that

$$\mathbf{A} = k\mathbf{B}.$$

It follows that

$$\alpha\mathbf{a}_x + 3\mathbf{a}_y - 2\mathbf{a}_z = 4k\mathbf{a}_x + k\beta\mathbf{a}_y + 8k\mathbf{a}_z,$$

Equating component by component yields the set of algebraic equations

$$\alpha = 4k; \qquad 3 = k\beta, \qquad 8k = -2. \tag{1}$$

It follows from Eq. (1) that

$$\alpha = -1, \qquad \beta = -12.$$

(b) $\mathbf{A}$ and $\mathbf{B}$ are **perpendicular**, implying that

$$\mathbf{A} \cdot \mathbf{B} = 0 \tag{2}$$

It follows at once from Eq. (2) and the definition of scalar product in terms of vector components that

$$4\alpha + 3\beta - 16 = 0. \tag{3}$$

In other words, one can only say that possible values of α and β must satisfy Eq. (3).

Problem 3

Express unit vectors in the Cartesian coordinates in terms of cylindrical coordinate unit vectors.

Solution

Since $\mathbf{a}_\rho = \boldsymbol{\rho}/\rho$, we can simply express $\mathbf{a}_\rho = \mathbf{a}_x \cos\phi + \mathbf{a}_y \sin\phi$. Also $\mathbf{a}_z$ is the same in both coordinate systems. As $\mathbf{a}_\rho$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$ form a right-hand set, $\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$. It follows by a cyclic permutation that $\mathbf{a}_\phi = -(\mathbf{a}_\rho \times \mathbf{a}_z) = -(\mathbf{a}_x \times \mathbf{a}_z)\cos\phi - (\mathbf{a}_y \times \mathbf{a}_z)\sin\phi = -\mathbf{a}_x \sin\phi + \mathbf{a}_y \cos\phi$. Where we used the mutual orthogonality of $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$. Thus,

$$\mathbf{a}_\rho = \mathbf{a}_x \cos\phi + \mathbf{a}_y \sin\phi,$$

and

$$\mathbf{a}_\phi = -\mathbf{a}_x \sin\phi + \mathbf{a}_y \cos\phi.$$

Solving the last two equations for $\mathbf{a}_x$ and $\mathbf{a}_y$, we arrive at

$$\mathbf{a}_x = \mathbf{a}_\rho \cos\phi - \mathbf{a}_\phi \sin\phi,$$

and

$$\mathbf{a}_y = \mathbf{a}_\rho \sin\phi + \mathbf{a}_\phi \cos\phi.$$

Problem 4

Express unit vectors in the spherical coordinates in terms of the Cartesian coordinate system unit vectors.

Solution

Recall that $\mathbf{a}_r = \mathbf{r}/r = \mathbf{a}_x \sin\theta \cos\phi + \mathbf{a}_y \sin\theta \sin\phi + \mathbf{a}_z \cos\theta$. We have also just determined $\mathbf{a}_\phi$, namely

$$\mathbf{a}_\phi = -\mathbf{a}_x \sin\phi + \mathbf{a}_y \cos\phi.$$

The mutual orthogonality of $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ dictates

$$\mathbf{a}_\theta = -(\mathbf{a}_r \times \mathbf{a}_\phi) = \mathbf{a}_\phi \times \mathbf{a}_r.$$

Using the mutual orthogonality of $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$, we obtain after little algebra,

$$\mathbf{a}_\theta = \mathbf{a}_x \cos\phi \cos\theta + \mathbf{a}_y \sin\phi \cos\theta - \mathbf{a}_z \sin\theta.$$